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On the Real Meaning of Bell's Theorem

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In the last couple of years many important results have been derived showing that Bell's inequalities are nothing else but the indicator of whether certain events and their probabilities can be represented within a Kolmogorovian probabilistic model. It has become evident that one can derive Bell's inequalities without mentioning locality, causality, hidden variables, etc. Many authors jumped to the conclusion that the original content of Bell's theorem had lost its meaning. I reconsider the original problem posed by Bell and I show that Bell's theorem is still valid.

1. INTRODUCTION

There were different derivations of Bell's inequalities till the late 1970s (Clauser and Shimony, 1978), each using certain assumptions about a hidden parameter, locality, and causation. These derivations and the violation of Bell's inequalities led to the conclusion that a local (deterministic or stochastic) hidden variable explanation of the EPR correlations contradicts quantum mechanics as well as the experimental results. Later, probability-theoretic investigations focused on the non-Kolmogorovian features of quantum probability (Accardi, 1984, 1988; De Muynck, 1986; Pitowsky, 1989; Beltrametti and Maczynski, 1991) discovered that Bell's inequalities are particular types of inequalities indicating whether certain events and their probabilities (which can be thought of as empirically given) can be represented or not within a Kolmogorovian probabilistic model. That is, the violation of Bell's inequalities itself has nothing to do with hidden parameters, locality, causality, etc. It is a quite common suggestion of this quantum probabilistic school that the original form of Bell's theorem loses

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its meaning. I am going to show in this paper that the situation is much more complex and Bell's theorem is still valid.

2. BELL INEQUALITIES AS TEST OF THE KOLMOGOROVIAN CHARACTER OF A PROBABILITY MODEL

First, recall a theorem (Pitowsky, 1989) illustrating that Bell's inequalities are equivalent with the Kolmogorovian character of a probability model. We need some notations and definitions for it:

Let S be a set of pairs of integers

$$S \subseteq \{\{i, j\} \mid 1 \le i < j \le n\}$$

Denote by $R(n, S)$ the real space of vectors like $(f_1, f_2, \ldots, f_n, \ldots, f_{ij}, \ldots)$.

For each $\varepsilon \in \{0, 1\}^n$ let u^ε be the following vector in $R(n, S)$:

$$u_i^\varepsilon = \varepsilon_i, \qquad 1 \le i \le n$$

$$u_{ij}^\varepsilon = \varepsilon_i \varepsilon_j, \qquad \{i, j\} \in S$$

The classical correlation polytope $C(n, S)$ is the closed convex hull in $R(n, S)$ of vectors $\{u^\varepsilon\}_{\varepsilon \in \{0,1\}^n}$ (see Fig. 1). That is,

$$C(n, S) := \left\{ a \in R(n, S) \,\middle|\, a = \sum_{\varepsilon \in \{0,1\}^n} \lambda_\varepsilon u^\varepsilon,\ \lambda_\varepsilon \ge 0, \sum \lambda_\varepsilon = 1 \right\}$$

Let $\bar{p} = (p_1, \ldots, p_n, \ldots, p_{ij}, \ldots) \in R(n, S)$. We say that $\bar{p}$ has a Kolmogorovian representation if there exists a Kolmogorovian probability space (Ω, Σ, μ) and (not necessarily distinct) events $A_1, A_2, \ldots, A_n \in \Sigma$ such that

$$p_i = \mu(A_i), \qquad 1 \le i \le n$$

$$p_{ij} = \mu(A_i \cap A_j), \qquad \{i, j\} \in S$$

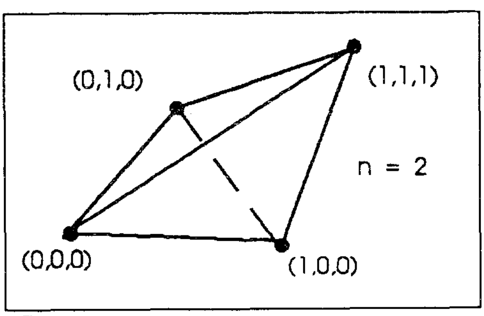


Fig. 1

Theorem 1 (Pitowsky, 1989). A correlation vector $\bar{p} = (p_1, \ldots, p_n, \ldots, p_{ij}, \ldots)$ has a Kolmogorovian representation if and only if $\bar{p} \in C(n, S)$.

In case $n = 3$ the condition $\bar{p} \in C(n, S)$ is nothing else but the well-known Bell inequalities

$$0 \leq p_{ij} \leq p_i \leq 1$$
$$0 \leq p_{ij} \leq p_j \leq 1$$
$$p_i + p_j - p_{ij} \leq 1$$
$$p_1 + p_2 + p_3 - p_{12} - p_{13} - p_{23} \leq 1$$
$$p_1 - p_{12} - p_{13} + p_{23} \geq 0$$
$$p_2 - p_{12} - p_{23} + p_{13} \geq 0$$
$$p_3 - p_{13} - p_{23} + p_{13} \geq 0$$

In case $n = 4$ and $S = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 4\}\}$ the condition $\bar{p} \in C(n, S)$ is equivalent with the Clauser–Horne inequalities.

In the EPR-type experiments, the measured probabilities (as well as those which are calculated from quantum mechanics) do not satisfy these conditions, therefore they do not have a Kolmogorovian representation.

3. BELL THEOREM > BELL INEQUALITIES

Because of the fact that the Bell inequalities can be derived without mentioning locality, causality, or hidden variables, there has arisen an opinion according to which the conditions of the original formulation of Bell's theorem have lost their meaning. But this is not true. Let us recall Bell's original idea.

If we encounter a correlation we can imagine two possibilities: (1) It is a direct correlation, that is, one event has a direct influence on the other (Fig. 2). (2) It is a common cause correlation, that is, there exists a third event in the common past having direct influence on both of them (Fig. 3).

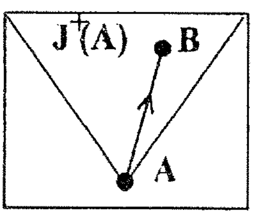


Fig. 2

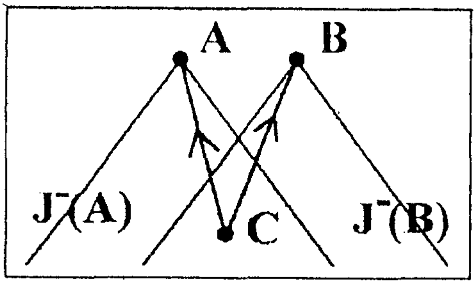


Fig. 3

Since events A and B are spatially separated, the direct correlation is excluded. This is typically the situation in the case of the (idealized) EPR experiment. The question is whether quantum mechanics is compatible with assumption of a common cause or not.

Bell investigated this question within the framework of a particular realization of a common cause mechanism described by a common parameter symbolizing all hereditary information from the common past, such that the measured probabilities are weighted averages of probabilities depending on this parameter. To make clear the realization of the original Bell theorem to the more recent results, we need the following three theorems:

Theorem 2. Let Λ be a parameter space with a normalized measure ρ. Let probabilities $\bar{\pi}(\lambda) = (\pi_1(\lambda), \ldots, \pi_n(\lambda), \ldots, \pi_{ij}(\lambda), \ldots)$ be dependent on the parameter $\lambda \in \Lambda$. Let probabilities p_i be constructed as

$$p_i = \int_\Lambda \pi_i(\lambda)\, d\rho(\lambda)$$

Then

$$(\forall\lambda)[\pi(\lambda) \in C(n, S)] \Rightarrow \bar{p} = (p_1, \ldots, p_n, \ldots, p_{ij}, \ldots) \in C(n, S)$$

Theorem 3. If a correlation vector $\bar{\pi} = (\pi_1, \ldots, \pi_n, \ldots, \pi_{ij}, \ldots)$ satisfies the independence condition

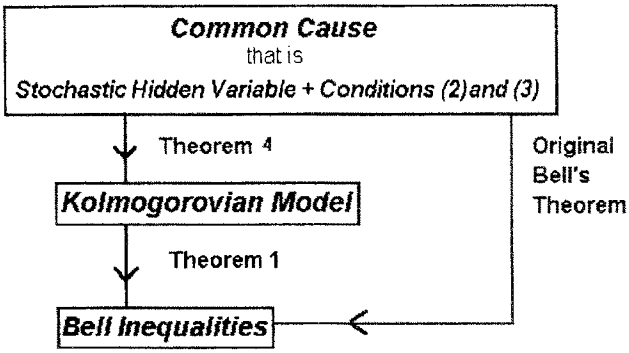
$$(\forall\{i, j\} \in S)[\pi_{ij} = \pi_i \cdot \pi_j]$$

then $\bar{\pi} \in C(n, S)$.

From Theorem 2 and Theorem 3 we have the following result.

Theorem 4. If probabilities $\bar{p} = (p_1, \ldots, p_n, \ldots, p_{ij}, \ldots)$ can be represented as

$$p_i = \int_\Lambda \pi_i(\lambda)\, d\rho(\lambda) \tag{1}$$



Scheme I

where probabilities $\bar{\pi} = (\pi_1, \ldots, \pi_n, \ldots, \pi_{ij}, \ldots)$ satisfy the independence condition

$$(\forall\lambda)(\forall\{i, j\} \in S)[\pi_{ij}(\lambda) = \pi_i(\lambda)\pi_j(\lambda)] \tag{2}$$

then $\bar{p} \in C(n, S)$.

What Bell's theorem states is exactly the same as Theorem 4 states (see Scheme I). Bell derived his inequalities from a particular realization of a common cause. It follows from the violation of these inequalities that Bell's particular representation is not possible.

4. CONCLUSIONS

We have seen that Bell's inequalities are the test of the Kolmogorovian representability. Bell's theorem belongs to a particular form of a common cause mechanism described by a common parameter symbolizing all hereditary information from the common past, such that the measured probabilities are weighted averages of probabilities depending on this parameter. These parameter-dependent probabilities are not supposed to be Kolmogorovian, but the independence condition makes them Kolmogorovian (Theorem 3). In opposition to Accardi (1988), I believe the independence condition is not "completely irrelevant for the proof" of Bell's theorem.

The restrictive character of Bell's concrete realization of a common cause mechanism consists in the assumption that the common cause events form a classical Boolean sublattice of events with a Kolmogorovian probability measure. That is why his investigation into the problem of causation is so tightly intertwined with the problem of hidden variables. Of course, there is no reason to suppose that a collection of quantum mechanical events form a Boolean sublattice as well as no reason to suppose that their probabilities could be described by a Kolmogorovian probability measure. Instead of this restrictive framework a more general realization of the common cause mechanism would be needed.

5. CONJECTURE

In order to strengthen the Bell theorem one has to find a more general model of a common cause and to prove that it also contradicts quantum mechanics. I would like finally to give such a very general description of the common cause mechanism.

Suppose $F_1, F_2, \ldots, F_m$ are common cause events of events $A_1, \ldots, A_n$ (Fig. 4). The minimal assumption about a common cause mechanism is that events $F_1, F_2, \ldots, F_m$ screen off events $A_1, \ldots, A_n$, that is,

$$(\forall F_k)(\forall\{i,j\}\in S)[p(A_i \wedge A_j)\,|F_k = p(A_i|F_k)p(A_j|F_k)]$$

The quantum mechanical conditioning generates a map over the correlation vectors:

$$\begin{array}{ccc} W & \xrightarrow{\text{Conditioning by } F_k} & W' \\ {\scriptstyle p_i = \mathrm{tr}(A_i W)}\downarrow & & \downarrow{\scriptstyle p'_i = \mathrm{tr}(A_i W')} \\ \bar{p}\in R(n, S) & \longrightarrow & \bar{p}'\in R(n, S) \end{array}$$

My conjecture is that

$$(\forall\{i,j\}\in S)[p'_{ij} = p'_i p'_j] \;\Rightarrow\; \bar{p}\in C(n, S)$$

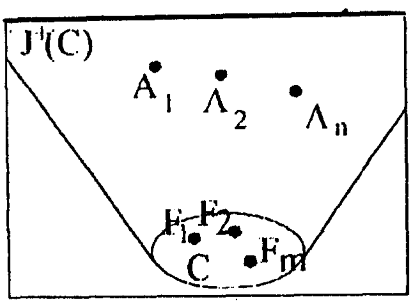


Fig. 4

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